

CLASSICAL 2-ABSORBING SUBMODULES OF MODULES OVER COMMUTATIVE RINGS

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ABSTRACT. In this article, all rings are commutative with nonzero identity. Let M be an R -module. A proper submodule N of M is called a *classical prime submodule*, if for each $m \in M$ and elements $a, b \in R$, $abm \in N$ implies that $am \in N$ or $bm \in N$. We introduce the concept of “classical 2-absorbing submodules” as a generalization of “classical prime submodules”. We say that a proper submodule N of M is a *classical 2-absorbing submodule* if whenever $a, b, c \in R$ and $m \in M$ with $abcm \in N$, then $abm \in N$ or $acm \in N$ or $bcm \in N$.

1. INTRODUCTION

Throughout this paper, we assume that all rings are commutative with $1 \neq 0$. Let R be a commutative ring and M be an R -module. A proper submodule N of M is said to be a *prime submodule*, if for each element $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M) = \{r \in R \mid rM \subseteq N\}$. A proper submodule N of M is called a *classical prime submodule*, if for each $m \in M$ and $a, b \in R$, $abm \in N$ implies that $am \in N$ or $bm \in N$. This notion of classical prime submodules has been extensively studied by Behboodi in [9, 10] (see also, [11], in which, the notion of “weakly prime submodules” is investigated). For more information on weakly prime submodules, the reader is referred to [3, 4, 12].

Badawi gave a generalization of prime ideals in [5] and said such ideals 2-absorbing ideals. A proper ideal I of R is a *2-absorbing ideal of R* if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. He proved that I is a 2-absorbing ideal of R if and only if whenever I_1, I_2, I_3 are ideals of R with $I_1 I_2 I_3 \subseteq I$, then $I_1 I_2 \subseteq I$ or $I_1 I_3 \subseteq I$ or $I_2 I_3 \subseteq I$. Anderson and Badawi [2] generalized the notion of 2-absorbing ideals to n -absorbing ideals. A proper ideal I of R is called an *n -absorbing* (resp. a *strongly n -absorbing*) *ideal* if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \dots, x_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals I_1, \dots, I_{n+1} of R), then there are n of the x_i 's (resp. n of the I_i 's) whose product is in I . The reader is referred to [6, 7, 8] for more concepts related to 2-absorbing ideals. Yousefian Darani and Soheilnia in [17] extended 2-absorbing ideals to 2-absorbing submodules. A proper submodule N of M is called a *2-absorbing submodule of M* if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. Generally, a proper submodule N of M is called an *n -absorbing submodule* if whenever $a_1 \cdots a_n m \in N$ for $a_1, \dots, a_n \in R$ and $m \in M$, then either $a_1 \cdots a_n \in (N :_R M)$ or there are $n - 1$ of a_i 's whose product with m is in N , see [18]. Several authors investigated properties of 2-absorbing submodules, for example [13, 14].

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In this paper we introduce the definition of classical 2-absorbing submodules. A proper submodule N of an R -module M is called *classical 2-absorbing submodule* if whenever $a, b, c \in R$ and $m \in M$ with $abcm \in N$, then $abm \in N$ or $acm \in N$ or $bcm \in N$. Clearly, every classical prime submodule is a classical 2-absorbing submodule. We show that every Noetherian R -module M contains a finite number of minimal classical 2-absorbing submodules [Theorem 2.10]. Further, we give the relationship between classical 2-absorbing submodules, classical prime submodules and 2-absorbing submodules [Proposition 2.5, Proposition 2.17]. Moreover, we characterize classical 2-absorbing submodules in [Theorem 2.7, Theorem 2.14]. In [Theorem 2.25, Theorem 2.27] we investigate classical 2-absorbing submodules of a finite direct product of modules.

2. CHARACTERIZATIONS OF CLASSICAL 2-ABSORBING SUBMODULES

First of all we give a module which has no classical 2-absorbing submodule.

Example 2.1. Let p be a fixed prime integer and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then

$$E(p) := \left\{ \alpha \in \mathbb{Q}/\mathbb{Z} \mid \alpha = \frac{r}{p^n} + \mathbb{Z} \text{ for some } r \in \mathbb{Z} \text{ and } n \in \mathbb{N}_0 \right\}$$

is a nonzero submodule of the \mathbb{Z} -module \mathbb{Q}/\mathbb{Z} . For each $t \in \mathbb{N}_0$, set

$$G_t := \left\{ \alpha \in \mathbb{Q}/\mathbb{Z} \mid \alpha = \frac{r}{p^t} + \mathbb{Z} \text{ for some } r \in \mathbb{Z} \right\}.$$

Notice that for each $t \in \mathbb{N}_0$, G_t is a submodule of $E(p)$ generated by $\frac{1}{p^t} + \mathbb{Z}$ for each $t \in \mathbb{N}_0$. Each proper submodule of $E(p)$ is equal to G_i for some $i \in \mathbb{N}_0$ (see, [16, Example 7.10]). However, no G_t is a classical 2-absorbing submodule of $E(p)$. Indeed, $\frac{1}{p^{t+3}} + \mathbb{Z} \in E(p)$. Then $p^3 \left(\frac{1}{p^{t+3}} + \mathbb{Z} \right) = \frac{1}{p^t} + \mathbb{Z} \in G_t$ but $p^2 \left(\frac{1}{p^{t+3}} + \mathbb{Z} \right) = \frac{1}{p^{t+1}} + \mathbb{Z} \notin G_t$.

Theorem 2.2. Let $f : M \rightarrow M'$ be an epimorphism of R -modules.

- (1) If N' is a classical 2-absorbing submodule of M' , then $f^{-1}(N')$ is a classical 2-absorbing submodule of M .
- (2) If N is a classical 2-absorbing submodule of M containing $\text{Ker}(f)$, then $f(N)$ is a classical 2-absorbing submodule of M' .

Proof. (1) Since f is epimorphism, $f^{-1}(N')$ is a proper submodule of M . Let $a, b, c \in R$ and $m \in M$ such that $abcm \in f^{-1}(N')$. Then $abcf(m) \in N'$. Hence $abf(m) \in N'$ or $acf(m) \in N'$ or $bcf(m) \in N'$, and thus $abm \in f^{-1}(N')$ or $acm \in f^{-1}(N')$ or $bcm \in f^{-1}(N')$. So, $f^{-1}(N')$ is a classical 2-absorbing submodule of M .

(2) Let $a, b, c \in R$ and $m' \in M'$ be such that $abcm' \in f(N)$. By assumption there exists $m \in M$ such that $m' = f(m)$ and so $f(abcm) \in f(N)$. Since $\text{Ker}(f) \subseteq N$, we have $abcm \in N$. It implies that $abm \in N$ or $acm \in N$ or $bcm \in N$. Hence $abm' \in f(N)$ or $acm' \in f(N)$ or $bcm' \in f(N)$. Consequently $f(N)$ is a classical 2-absorbing submodule of M' . \square

As an immediate consequence of Theorem 2.2 we have the following corollary.

Corollary 2.3. *Let M be an R -module and $L \subseteq N$ be submodules of M . Then N is a classical 2-absorbing submodule of M if and only if N/L is a classical 2-absorbing submodule of M/L .*

Proposition 2.4. *Let M be an R -module and N_1, N_2 be classical prime submodules of M . Then $N_1 \cap N_2$ is a classical 2-absorbing submodule of M .*

Proof. Let for some $a, b, c \in R$ and $m \in M$, $abcm \in N_1 \cap N_2$. Since N_1 is a classical prime submodule, then we may assume that $am \in N_1$. Likewise, assume that $bm \in N_2$. Hence $abm \in N_1 \cap N_2$ which implies $N_1 \cap N_2$ is a classical 2-absorbing submodule. \square

Proposition 2.5. *Let N be a proper submodule of an R -module M .*

- (1) *If N is a 2-absorbing submodule of M , then N is a classical 2-absorbing submodule of M .*
- (2) *N is a classical prime submodule of M if and only if N is a 2-absorbing submodule of M and $(N :_R M)$ is a prime ideal of R .*

Proof. (1) Assume that N is a 2-absorbing submodule of M . Let $a, b, c \in R$ and $m \in M$ such that $abcm \in N$. Therefore either $acm \in N$ or $bcm \in N$ or $ab \in (N : M)$. The first two cases lead us to the claim. In the third case we have that $abm \in N$. Consequently N is a classical 2-absorbing submodule.

(2) It is evident that if N is classical prime, then it is 2-absorbing. Also, [3, Lemma 2.1] implies that $(N :_R M)$ is a prime ideal of R . Assume that N is a 2-absorbing submodule of M and $(N :_R M)$ is a prime ideal of R . Let $abm \in N$ for some $a, b \in R$ and $m \in M$ such that neither $am \in N$ nor $bm \in N$. Then $ab \in (N :_R M)$ and so either $a \in (N :_R M)$ or $b \in (N :_R M)$. This contradiction shows that N is classical prime. \square

The following example shows that the converse of Proposition 2.5(1) is not true.

Example 2.6. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Q}$ where p, q are two distinct prime integers. One can easily see that the zero submodule of M is a classical 2-absorbing submodule. Notice that $pq(1, 1, 0) = (0, 0, 0)$, but $p(1, 1, 0) \neq (0, 0, 0)$, $q(1, 1, 0) \neq (0, 0, 0)$ and $pq(1, 1, 1) \neq 0$. So the zero submodule of M is not 2-absorbing. Also, part (2) of Proposition 2.5 shows that the zero submodule is not a classical prime submodule. Hence the two concepts of classical prime submodules and of classical 2-absorbing submodules are different in general.

Let M be an R -module and N a submodule of M . For every $a \in R$, $\{m \in M \mid am \in N\}$ is denoted by $(N :_R a)$. It is easy to see that $(N :_M a)$ is a submodule of M containing N .

Theorem 2.7. *Let M be an R -module and N be a proper submodule of M . The following conditions are equivalent:*

- (1) *N is classical 2-absorbing;*
- (2) *For every $a, b, c \in R$, $(N :_M abc) = (N :_M ab) \cup (N :_M ac) \cup (N :_M bc)$;*
- (3) *For every $a, b \in R$ and $m \in M$ with $abm \notin N$, $(N :_R abm) = (N :_R am) \cup (N :_R bm)$;*
- (4) *For every $a, b \in R$ and $m \in M$ with $abm \notin N$, $(N :_R abm) = (N :_R am)$ or $(N :_R abm) = (N :_R bm)$;*
- (5) *For every $a, b \in R$ and every ideal I of R and $m \in M$ with $abIm \subseteq N$, either $abm \in N$ or $aIm \subseteq N$ or $bIm \subseteq N$;*

- (6) For every $a \in R$ and every ideal I of R and $m \in M$ with $aIm \not\subseteq N$, $(N :_R aIm) = (N :_R am)$ or $(N :_R aIm) = (N :_R Im)$;
- (7) For every $a \in R$ and every ideals I, J of R and $m \in M$ with $aIJm \subseteq N$, either $aIm \subseteq N$ or $aJm \subseteq N$ or $IJm \subseteq N$;
- (8) For every ideals I, J of R and $m \in M$ with $IJm \not\subseteq N$, $(N :_R IJm) = (N :_R Im)$ or $(N :_R IJm) = (N :_R Jm)$;
- (9) For every ideals I, J, K of R and $m \in M$ with $IJKm \subseteq N$, either $IJm \subseteq N$ or $IKm \subseteq N$ or $JKm \subseteq N$;
- (10) For every $m \in M \setminus N$, $(N :_R m)$ is a 2-absorbing ideal of R .

Proof. (1) \Rightarrow (2) Suppose that N is a classical 2-absorbing submodule of M . Let $m \in (N :_M abc)$. Then $abcm \in N$. Hence $abm \in N$ or $acm \in N$ or $bcm \in N$. Therefore $m \in (N :_M ab)$ or $m \in (N :_M ac)$ or $m \in (N :_M bc)$. Consequently, $(N :_M abc) = (N :_M ab) \cup (N :_M ac) \cup (N :_M bc)$.

(2) \Rightarrow (3) Let $abm \notin N$ for some $a, b \in R$ and $m \in M$. Assume that $x \in (N :_R abm)$. Then $abxm \in N$, and so $m \in (N :_M abx)$. Since $abm \notin N$, $m \notin (N :_M ab)$. Thus by part (1), $m \in (N :_M ax)$ or $m \in (N :_M bx)$, whence $x \in (N :_R am)$ or $x \in (N :_R bm)$. Therefore $(N :_R abm) = (N :_R am) \cup (N :_R bm)$.

(3) \Rightarrow (4) By the fact that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them.

(4) \Rightarrow (5) Let for some $a, b \in R$, an ideal I of R and $m \in M$, $abIm \subseteq N$. Hence $I \subseteq (N :_R abm)$. If $abm \in N$, then we are done. Assume that $abm \notin N$. Therefore by part (4) we have that $I \subseteq (N :_R am)$ or $I \subseteq (N :_R bm)$, i.e., $aIm \subseteq N$ or $bIm \subseteq N$.

(5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8) \Rightarrow (9) Have proofs similar to that of the previous implications.

(9) \Rightarrow (1) Is trivial.

(9) \Leftrightarrow (10) Straightforward. \square

Corollary 2.8. Let R be a ring and I be a proper ideal of R .

- (1) $_RI$ is a classical 2-absorbing submodule of R if and only if I is a 2-absorbing ideal of R .
- (2) Every proper ideal of R is 2-absorbing if and only if for every R -module M and every proper submodule N of M , N is a classical 2-absorbing submodule of M .

Proof. (1) Let I be a classical 2-absorbing submodule of R . Then by Theorem 2.7, $(I :_R 1) = I$ is a 2-absorbing ideal of R . For the converse see part (1) of Proposition 2.5.

(2) Assume that every proper ideal of R is 2-absorbing. Let N be a proper submodule of an R -module M . Since for every $m \in M \setminus N$, $(N :_R m)$ is a proper ideal of R , then it is a 2-absorbing ideal of R . Hence by Theorem 2.7, N is a classical 2-absorbing submodule of M . We have the converse immediately by part (1). \square

Proposition 2.9. Let M be an R -module and $\{K_i \mid i \in I\}$ be a chain of classical 2-absorbing submodules of M . Then $\cap_{i \in I} K_i$ is a classical 2-absorbing submodule of M .

Proof. Suppose that $abcm \in \cap_{i \in I} K_i$ for some $a, b, c \in R$ and $m \in M$. Assume that $abm \notin \cap_{i \in I} K_i$ and $acm \notin \cap_{i \in I} K_i$. Then there are $t, l \in I$ where $abm \notin K_t$ and $acm \notin K_l$. Hence, for every $K_s \subseteq K_t$ and every $K_d \subseteq K_l$ we have that $abm \notin K_s$

and $acm \notin K_d$. Thus, for every submodule K_h such that $K_h \subseteq K_t$ and $K_h \subseteq K_l$ we get $bcm \in K_h$. Hence $bcm \in \cap_{i \in I} K_i$. \square

A classical 2-absorbing submodule of M is called *minimal*, if for any classical 2-absorbing submodule K of M such that $K \subseteq N$, then $K = N$. Let L be a classical 2-absorbing submodule of M . Set

$$\Gamma = \{K \mid K \text{ is a classical 2-absorbing submodule of } M \text{ and } K \subseteq L\}.$$

If $\{K_i : i \in I\}$ is any chain in Γ , then $\cap_{i \in I} K_i$ is in Γ , by Proposition 2.9. By Zorn's Lemma, Γ contains a minimal member which is clearly a minimal classical 2-absorbing submodule of M . Thus, every classical 2-absorbing submodule of M contains a minimal classical 2-absorbing submodule of M . If M is a finitely generated, then it is clear that M contains a minimal classical 2-absorbing submodule.

Theorem 2.10. *Let M be a Noetherian R -module. Then M contains a finite number of minimal classical 2-absorbing submodules.*

Proof. Suppose that the result is false. Let Γ denote the collection of proper submodules N of M such that the module M/N has an infinite number of minimal classical 2-absorbing submodules. Since $0 \in \Gamma$ we get $\Gamma \neq \emptyset$. Therefore Γ has a maximal member T , since M is a Noetherian R -module. It is clear that T is not a classical 2-absorbing submodule. Therefore, there exists an element $m \in M \setminus T$ and ideals I, J, K in R such that $IJKm \subseteq T$ but $IJm \not\subseteq T$, $IKm \not\subseteq T$ and $JKm \not\subseteq T$. The maximality of T implies that $M/(T + IJm)$, $M/(T + IKm)$ and $M/(T + JKm)$ have only finitely many minimal classical 2-absorbing submodules. Suppose P/T be a minimal classical 2-absorbing submodule of M/T . So $IJKm \subseteq T \subseteq P$, which implies that $IJm \subseteq P$ or $IKm \subseteq P$ or $JKm \subseteq P$. Thus $P/(T + IJm)$ is a minimal classical 2-absorbing submodule of $M/(T + IJm)$ or $P/(T + IKm)$ is a minimal classical 2-absorbing submodule of $M/(T + IKm)$ or $P/(T + JKm)$ is a minimal classical 2-absorbing submodule of $M/(T + JKm)$. Thus, there are only a finite number of possibilities for the submodule P . This is a contradiction. \square

We recall from [5] that if I is a 2-absorbing ideal of a ring R , then either $\sqrt{I} = P$ where P is a prime ideal of R or $\sqrt{I} = P_1 \cap P_2$ where P_1, P_2 are the only distinct minimal prime ideals of I .

Corollary 2.11. *Let N be a classical 2-absorbing submodule of an R -module M . Suppose that $m \in M \setminus N$ and $\sqrt{(N :_R m)} = P$ where P is a prime ideal of R and $(N :_R m) \neq P$. Then for each $x \in \sqrt{(N :_R m)} \setminus (N :_R m)$, $(N :_R xm)$ is a prime ideal of R containing P . Furthermore, either $(N :_R xm) \subseteq (N :_R ym)$ or $(N :_R ym) \subseteq (N :_R xm)$ for every $x, y \in \sqrt{(N :_R m)} \setminus (N :_R m)$.*

Proof. By Theorem 2.7 and [5, Theorem 2.5]. \square

Corollary 2.12. *Let N be a classical 2-absorbing submodule of an R -module M . Suppose that $m \in M \setminus N$ and $\sqrt{(N :_R m)} = P_1 \cap P_2$ where P_1 and P_2 are the only nonzero distinct prime ideals of R that are minimal over $(N :_R m)$. Then for each $x \in \sqrt{(N :_R m)} \setminus (N :_R m)$, $(N :_R xm)$ is a prime ideal of R containing P_1 and P_2 . Furthermore, either $(N :_R xm) \subseteq (N :_R ym)$ or $(N :_R ym) \subseteq (N :_R xm)$ for every $x, y \in \sqrt{(N :_R m)} \setminus (N :_R m)$.*

Proof. By Theorem 2.7 and [5, Theorem 2.6]. \square

An R -module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R . Let N and K be submodules of a multiplication R -module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R . The product of N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [1, Theorem 3.4], the product of N and K is independent of presentations of N and K .

Proposition 2.13. *Let M be a multiplication R -module and N be a proper submodule of M . The following conditions are equivalent:*

- (1) N is a classical 2-absorbing submodule of M ;
- (2) If $N_1N_2N_3m \subseteq N$ for some submodules N_1, N_2, N_3 of M and $m \in M$, then either $N_1N_2m \subseteq N$ or $N_1N_3m \subseteq N$ or $N_2N_3m \subseteq N$.

Proof. (1) \Rightarrow (2) Let $N_1N_2N_3m \subseteq N$ for some submodules N_1, N_2, N_3 of M and $m \in M$. Since M is multiplication, there are ideals I_1, I_2, I_3 of R such that $N_1 = I_1M$, $N_2 = I_2M$ and $N_3 = I_3M$. Therefore $I_1I_2I_3m \subseteq N$, and so either $I_1I_2m \subseteq N$ or $I_1I_3m \subseteq N$ or $I_2I_3m \subseteq N$. Hence $N_1N_2m \subseteq N$ or $N_1N_3m \subseteq N$ or $N_2N_3m \subseteq N$. (2) \Rightarrow (1) Suppose that $I_1I_2I_3m \subseteq N$ for some ideals I_1, I_2, I_3 of R and some $m \in M$. It is sufficient to set $N_1 := I_1M$, $N_2 := I_2M$ and $N_3 = I_3M$ in part (2). \square

In [15], Quattararo et al. said that a commutative ring R is a u -ring provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a um -ring is a ring R with the property that an R -module which is equal to a finite union of submodules must be equal to one of them. They show that every Bézout ring is a u -ring. Moreover, they proved that every Prüfer domain is a u -domain. Also, any ring which contains an infinite field as a subring is a u -ring, [16, Exercise 3.63].

Theorem 2.14. *Let R be a um -ring, M be an R -module and N be a proper submodule of M . The following conditions are equivalent:*

- (1) N is classical 2-absorbing;
- (2) For every $a, b, c \in R$, $(N :_M abc) = (N :_M ab)$ or $(N :_M abc) = (N :_M ac)$ or $(N :_M abc) = (N :_M bc)$;
- (3) For every $a, b, c \in R$ and every submodule L of M , $abcL \subseteq N$ implies that $abL \subseteq N$ or $acL \subseteq N$ or $bcL \subseteq N$;
- (4) For every $a, b \in R$ and every submodule L of M with $abL \not\subseteq N$, $(N :_R abL) = (N :_R aL)$ or $(N :_R abL) = (N :_R bL)$;
- (5) For every $a, b \in R$, every ideal I of R and every submodule L of M , $abIL \subseteq N$ implies that $abL \subseteq N$ or $aIL \subseteq N$ or $bIL \subseteq N$;
- (6) For every $a \in R$, every ideal I of R and every submodule L of M with $aIL \not\subseteq N$, $(N :_R aIL) = (N :_R aL)$ or $(N :_R aIL) = (N :_R IL)$;
- (7) For every $a \in R$, every ideals I, J of R and every submodule L of M , $aIJL \subseteq N$ implies that $aIL \subseteq N$ or $aJL \subseteq N$ or $IJL \subseteq N$;
- (8) For every ideals I, J of R and every submodule L of M with $IJL \not\subseteq N$, $(N :_R IJL) = (N :_R IL)$ or $(N :_R IJL) = (N :_R JL)$;
- (9) For every ideals I, J, K of R and every submodule L of M , $IJKL \subseteq N$ implies that $IJL \subseteq N$ or $IKL \subseteq N$ or $JKL \subseteq N$;
- (10) For every submodule L of M not contained in N , $(N :_R L)$ is a 2-absorbing ideal of R .

Proof. Similar to the proof of Theorem 2.7. \square

Proposition 2.15. *Let R be a um-ring and N be a proper submodule of an R -module M . Then N is a classical 2-absorbing submodule of M if and only if N is a 4-absorbing submodule of M and $(N :_R M)$ is a 2-absorbing ideal of R .*

Proof. It is trivial that if N is classical 2-absorbing, then it is 4-absorbing. Also, Theorem 2.14 implies that $(N :_R M)$ is a 2-absorbing ideal of R . Now, assume that N is a 4-absorbing submodule of M and $(N :_R M)$ is a 2-absorbing ideal of R . Let $a_1 a_2 a_3 m \in N$ for some $a_1, a_2, a_3 \in R$ and $m \in M$ such that neither $a_1 a_2 m \in N$ nor $a_1 a_3 m \in N$ nor $a_2 a_3 m \in N$. Then $a_1 a_2 a_3 \in (N :_R M)$ and so either $a_1 a_2 \in (N :_R M)$ or $a_1 a_3 \in (N :_R M)$ or $a_2 a_3 \in (N :_R M)$. This contradiction shows that N is classical 2-absorbing. \square

Proposition 2.16. *Let M be an R -module and N be a classical 2-absorbing submodule of M . The following conditions hold:*

- (1) *For every $a, b, c \in R$ and $m \in M$, $(N :_R abcm) = (N :_R abm) \cup (N :_R acm) \cup (N :_R bcm)$;*
- (2) *If R is a u-ring, then for every $a, b, c \in R$ and $m \in M$, $(N :_R abcm) = (N :_R abm)$ or $(N :_R acm) = (N :_R bcm)$.*

Proof. (1) Let $a, b, c \in R$ and $m \in M$. Suppose that $r \in (N :_R abcm)$. Then $abc(rm) \in N$. So, either $ab(rm) \in N$ or $ac(rm) \in N$ or $bc(rm) \in N$. Therefore, either $r \in (N :_R abm)$ or $r \in (N :_R acm)$ or $r \in (N :_R bcm)$. Consequently $(N :_R abcm) = (N :_R abm) \cup (N :_R acm) \cup (N :_R bcm)$.

(2) Use part (1). \square

Proposition 2.17. *Let R be a um-ring, M be a multiplication R -module and N be a proper submodule of M . The following conditions are equivalent:*

- (1) *N is a classical 2-absorbing submodule of M ;*
- (2) *If $N_1 N_2 N_3 N_4 \subseteq N$ for some submodules N_1, N_2, N_3, N_4 of M , then either $N_1 N_2 N_4 \subseteq N$ or $N_1 N_3 N_4 \subseteq N$ or $N_2 N_3 N_4 \subseteq N$;*
- (3) *If $N_1 N_2 N_3 \subseteq N$ for some submodules N_1, N_2, N_3 of M , then either $N_1 N_2 \subseteq N$ or $N_1 N_3 \subseteq N$ or $N_2 N_3 \subseteq N$;*
- (4) *N is a 2-absorbing submodule of M ;*
- (5) *$(N :_R M)$ is a 2-absorbing ideal of R .*

Proof. (1) \Rightarrow (2) Let $N_1 N_2 N_3 N_4 \subseteq N$ for some submodules N_1, N_2, N_3, N_4 of M . Since M is multiplication, there are ideals I_1, I_2, I_3 of R such that $N_1 = I_1 M$, $N_2 = I_2 M$ and $N_3 = I_3 M$. Therefore $I_1 I_2 I_3 N_4 \subseteq N$, and so $I_1 I_2 N_4 \subseteq N$ or $I_1 I_3 N_4 \subseteq N$ or $I_2 I_3 N_4 \subseteq N$. Thus by Theorem 2.14, either $N_1 N_2 N_4 \subseteq N$ or $N_1 N_3 N_4 \subseteq N$ or $N_2 N_3 N_4 \subseteq N$.

(2) \Rightarrow (3) Is easy.

(3) \Rightarrow (4) Suppose that $I_1 I_2 K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of M . It is sufficient to set $N_1 := I_1 M$, $N_2 := I_2 M$ and $N_3 = K$ in part (3).

(4) \Rightarrow (1) By part (1) of Proposition 2.5.

(4) \Rightarrow (5) By [14, Theorem 2.3].

(5) \Rightarrow (4) Let $I_1 I_2 K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of M . Since M is multiplication, then there is an ideal I_3 of R such that $K = I_3 M$. Hence $I_1 I_2 I_3 \subseteq (N :_R M)$ which implies that either $I_1 I_2 \subseteq (N :_R M)$ or $I_1 I_3 \subseteq (N :_R M)$ or $I_2 I_3 \subseteq (N :_R M)$. If $I_1 I_2 \subseteq (N :_R M)$, then we are done. So, suppose that

$I_1 I_3 \subseteq (N :_R M)$. Thus $I_1 I_3 M = I_1 K \subseteq N$. Similary if $I_2 I_3 \subseteq (N :_R M)$, then we have $I_2 K \subseteq N$. \square

Definition 2.18. Let R be a um-ring, M be an R -module and S be a subset of $M \setminus \{0\}$. If for all ideals I, J, Q of R and all submodules K, L of M , $(K + IJL) \cap S \neq \emptyset$ and $(K + IQL) \cap S \neq \emptyset$ and $(K + JQL) \cap S \neq \emptyset$ implies $(K + IJQL) \cap S \neq \emptyset$, then the subset S is called *classical 2-absorbing m-closed*.

Proposition 2.19. Let R be a um-ring, M be R -module and N a submodule of M . Then N is a classical 2-absorbing submodule if and only if $M \setminus N$ is a classical 2-absorbing m-closed.

Proof. Suppose that N is a classical 2-absorbing submodule of M and I, J, Q are ideals of R and K, L are submodules of M such that $(K + IJL) \cap S \neq \emptyset$ and $(K + IQL) \cap S \neq \emptyset$ and $(K + JQL) \cap S \neq \emptyset$ where $S = M \setminus N$. Assume that $(K + IJQL) \cap S = \emptyset$. Then $K + IJQL \subseteq N$ and so $K \subseteq N$ and $IJQL \subseteq N$. Since N is a classical 2-absorbing submodule, we get $IJL \subseteq N$ or $IQL \subseteq N$ or $JQL \subseteq N$. If $IJL \subseteq N$, then we get $(K + IJL) \cap S = \emptyset$, since $K \subseteq N$. This is a contradiction. By the other cases we get similar contradictions. Now for the converse suppose that $S = M \setminus N$ is a classical 2-absorbing m-closed and assume that $IJQL \subseteq N$ for some ideals I, J, Q of R and submodule L of M . Then we get for submodule $K = (0)$, $K + IJQL \subseteq N$. Thus $(K + IJQL) \cap S = \emptyset$. Since S is a classical 2-absorbing m-closed, $(K + IJL) \cap S = \emptyset$ or $(K + IQL) \cap S = \emptyset$ or $(K + JQL) \cap S = \emptyset$. Hence $IJL \subseteq N$ or $IQL \subseteq N$ or $JQL \subseteq N$. So N is a classical 2-absorbing submodule. \square

Proposition 2.20. Let R be a um-ring, M be an R -module, N a submodule of M and $S = M \setminus N$. The following conditions are equivalent:

- (1) N is a classical 2-absorbing submodule of M ;
- (2) S is a classical 2-absorbing m-closed;
- (3) For every ideals I, J, Q of R and every submodule L of M , if $IJL \cap S \neq \emptyset$ and $IQL \cap S \neq \emptyset$ and $JQL \cap S \neq \emptyset$, then $IJQL \cap S \neq \emptyset$;
- (4) For every ideals I, J, Q of R and every $m \in M$, if $IJm \cap S \neq \emptyset$ and $IQm \cap S \neq \emptyset$ and $JQm \cap S \neq \emptyset$, then $IJQm \cap S \neq \emptyset$.

Proof. It follows from the previous Proposition, Theorem 2.7 and Theorem 2.14. \square

Theorem 2.21. Let R be a um-ring, M be an R -module and S be a classical 2-absorbing m-closed. Then the set of all submodules of M which are disjoint from S has at least one maximal element. Any such maximal element is a classical 2-absorbing submodule.

Proof. Let $\Psi = \{N \mid N \text{ is a submodule of } M \text{ and } N \cap S = \emptyset\}$. Then $(0) \in \Psi \neq \emptyset$. Since Ψ is partially ordered by using Zorn's Lemma we get at least a maximal element of Ψ , say P , with property $P \cap S = \emptyset$. Now we will show that P is classical 2-absorbing. Suppose that $IJQL \subseteq P$ for ideals I, J, Q of R and submodule L of M . Assume that $IJL \not\subseteq P$ or $IQL \not\subseteq P$ or $JQL \not\subseteq P$. Then by the maximality of P we get $(IJL + P) \cap S \neq \emptyset$ and $(IQL + P) \cap S \neq \emptyset$ and $(JQL + P) \cap S \neq \emptyset$. Since S is a classical 2-absorbing m-closed we have $(IJQL + P) \cap S \neq \emptyset$. Hence $P \cap S \neq \emptyset$, which is a contradiction. Thus P is a classical 2-absorbing submodule of M . \square

Theorem 2.22. Let R be a um-ring and M be an R -module.

- (1) If F is a flat R -module and N is a classical 2-absorbing submodule of M such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a classical 2-absorbing submodule of $F \otimes M$.
- (2) Suppose that F is a faithfully flat R -module. Then N is a classical 2-absorbing submodule of M if and only if $F \otimes N$ is a classical 2-absorbing submodule of $F \otimes M$.

Proof. (1) Let $a, b, c \in R$. Then we get by Theorem 2.14, $(N :_M abc) = (N :_M ab)$ or $(N :_M abc) = (N :_M ac)$ or $(N :_M abc) = (N :_M bc)$. Assume that $(N :_M abc) = (N :_M ab)$. Then by [4, Lemma 3.2], $(F \otimes N :_{F \otimes M} abc) = F \otimes (N :_M abc) = F \otimes (N :_M ab) = (F \otimes N :_{F \otimes M} ab)$. Again Theorem 2.14 implies that $F \otimes N$ is a classical 2-absorbing submodule of $F \otimes M$.

(2) Let N be a classical 2-absorbing submodule of M and assume that $F \otimes N = F \otimes M$. Then $0 \rightarrow F \otimes N \xrightarrow{\subseteq} F \otimes M \rightarrow 0$ is an exact sequence. Since F is a faithfully flat module, $0 \rightarrow N \xrightarrow{\subseteq} M \rightarrow 0$ is an exact sequence. So $N = M$, which is a contradiction. So $F \otimes N \neq F \otimes M$. Then $F \otimes N$ is a classical 2-absorbing submodule by (1). Now for conversely, let $F \otimes N$ be a classical 2-absorbing submodule of $F \otimes M$. We have $F \otimes N \neq F \otimes M$ and so $N \neq M$. Let $a, b, c \in R$. Then $(F \otimes N :_{F \otimes M} abc) = (F \otimes N :_{F \otimes M} ab)$ or $(F \otimes N :_{F \otimes M} abc) = (F \otimes N :_{F \otimes M} ac)$ or $(F \otimes N :_{F \otimes M} abc) = (F \otimes N :_{F \otimes M} bc)$ by Theorem 2.14. Assume that $(F \otimes N :_{F \otimes M} abc) = (F \otimes N :_{F \otimes M} ab)$. Hence $F \otimes (N :_M ab) = (F \otimes N :_{F \otimes M} ab) = (F \otimes N :_{F \otimes M} abc) = F \otimes (N :_M abc)$. So $0 \rightarrow F \otimes (N :_M ab) \xrightarrow{\subseteq} F \otimes (N :_M abc) \rightarrow 0$ is an exact sequence. Since F is a faithfully flat module, $0 \rightarrow (N :_M ab) \xrightarrow{\subseteq} (N :_M abc) \rightarrow 0$ is an exact sequence which implies that $(N :_M ab) = (N :_M abc)$. Consequently N is a classical 2-absorbing submodule of M by Theorem 2.14. \square

Corollary 2.23. *Let R be a um-ring, M be an R -module and X be an indeterminate. If N is a classical 2-absorbing submodule of M , then $N[X]$ is a classical 2-absorbing submodule of $M[X]$.*

Proof. Assume that N is a classical 2-absorbing submodule of M . Notice that $R[X]$ is a flat R -module. So by Theorem 2.22, $R[X] \otimes N \simeq N[X]$ is a classical 2-absorbing submodule of $R[X] \otimes M \simeq M[X]$. \square

For an R -module M , the set of zero-divisors of M is denoted by $Z_R(M)$.

Proposition 2.24. *Let M be an R -module, N be a submodule and S be a multiplicative subset of R .*

- (1) *If N is a classical 2-absorbing submodule of M such that $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a classical 2-absorbing submodule of $S^{-1}M$.*
- (2) *If $S^{-1}N$ is a classical 2-absorbing submodule of $S^{-1}M$ such that $Z_R(M/N) \cap S = \emptyset$, then N is a classical 2-absorbing submodule of M .*

Proof. (1) Let N be a classical 2-absorbing submodule of M and $(N :_R M) \cap S = \emptyset$. Suppose that $\frac{a_1 a_2 a_3 m}{s_1 s_2 s_3 s_4} \in S^{-1}N$. Then there exist $n \in N$ and $s \in S$ such that $\frac{a_1 a_2 a_3 m}{s_1 s_2 s_3 s_4} = \frac{n}{s}$. Therefore there exists an $s' \in S$ such that $s' s a_1 a_2 a_3 m = s' s_1 s_2 s_3 s_4 n \in N$. So $a_1 a_2 a_3 (s^* m) \in N$ for $s^* = s' s$. Since N is a classical 2-absorbing submodule we get $a_1 a_2 (s^* m) \in N$ or $a_1 a_3 (s^* m) \in N$ or $a_2 a_3 (s^* m) \in N$. Thus $\frac{a_1 a_2 m}{s_1 s_2 s_4} = \frac{a_1 a_2 (s^* m)}{s_1 s_2 s_4 s^*} \in S^{-1}N$ or $\frac{a_1 a_3 m}{s_1 s_3 s_4} \in S^{-1}N$ or $\frac{a_2 a_3 m}{s_2 s_3 s_4} \in S^{-1}N$.

(2) Assume that $S^{-1}N$ is a classical 2-absorbing submodule of $S^{-1}M$ and $Z_R(M/N) \cap S = \emptyset$. Let $a, b, c \in R$ and $m \in M$ such that $abcm \in N$. Then $\frac{a}{1} \frac{b}{1} \frac{c}{1} \frac{m}{1} \in S^{-1}N$. Therefore $\frac{a}{1} \frac{b}{1} \frac{m}{1} \in S^{-1}N$ or $\frac{a}{1} \frac{c}{1} \frac{m}{1} \in S^{-1}N$ or $\frac{b}{1} \frac{c}{1} \frac{m}{1} \in S^{-1}N$. We may assume that $\frac{a}{1} \frac{b}{1} \frac{m}{1} \in S^{-1}N$. So there exists $u \in S$ such that $uabm \in N$. But $Z_R(M/N) \cap S = \emptyset$, whence $abm \in N$. Consequently N is a classical 2-absorbing submodule of M . \square

Let R_i be a commutative ring with identity and M_i be an R_i -module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R -module and each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2 .

Theorem 2.25. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R -module where M_1 is an R_1 -module and M_2 is an R_2 -module. Suppose that $N = N_1 \times N_2$ is a proper submodule of M . Then the following conditions are equivalent:*

- (1) N is a classical 2-absorbing submodule of M ;
- (2) Either $N_1 = M_1$ and N_2 is a classical 2-absorbing submodule of M_2 or $N_2 = M_2$ and N_1 is a classical 2-absorbing submodule of M_1 or N_1, N_2 are classical prime submodules of M_1, M_2 , respectively.

Proof. (1) \Rightarrow (2) Suppose that N is a classical 2-absorbing submodule of M such that $N_2 = M_2$. From our hypothesis, N is proper, so $N_1 \neq M_1$. Set $M' = \frac{M}{\{0\} \times M_2}$. Hence $N' = \frac{N}{\{0\} \times M_2}$ is a classical 2-absorbing submodule of M' by Corollary 2.3. Also observe that $M' \cong M_1$ and $N' \cong N_1$. Thus N_1 is a classical 2-absorbing submodule of M_1 . Suppose that $N_1 \neq M_1$ and $N_2 \neq M_2$. We show that N_1 is a classical prime submodule of M_1 . Since $N_2 \neq M_2$, there exists $m_2 \in M_2 \setminus N_2$. Let $abm_1 \in N_1$ for some $a, b \in R_1$ and $m_1 \in M_1$. Thus $(a, 1)(b, 1)(1, 0)(m_1, m_2) = (abm_1, 0) \in N = N_1 \times N_2$. So either $(a, 1)(1, 0)(m_1, m_2) = (am_1, 0) \in N$ or $(b, 1)(1, 0)(m_1, m_2) = (bm_1, 0) \in N$. Hence either $am_1 \in N_1$ or $bm_1 \in N_1$ which shows that N_1 is a classical prime submodule of M_1 . Similarly we can show that N_2 is a classical prime submodule of M_2 .

(2) \Rightarrow (1) Suppose that $N = N_1 \times M_2$ where N_1 is a classical 2-absorbing (resp. classical prime) submodule of M_1 . Then it is clear that N is a classical 2-absorbing (resp. classical prime) submodule of M . Now, assume that $N = N_1 \times N_2$ where N_1 and N_2 are classical prime submodules of M_1 and M_2 , respectively. Hence $(N_1 \times M_2) \cap (M_1 \times N_2) = N_1 \times N_2 = N$ is a classical 2-absorbing submodule of M , by Proposition 2.4. \square

Lemma 2.26. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a decomposable ring and $M = M_1 \times M_2 \times \cdots \times M_n$ be an R -module where for every $1 \leq i \leq n$, M_i is an R_i -module, respectively. A proper submodule N of M is a classical prime submodule of M if and only if $N = \times_{i=1}^n N_i$ such that for some $k \in \{1, 2, \dots, n\}$, N_k is a classical prime submodule of M_k , and $N_i = M_i$ for every $i \in \{1, 2, \dots, n\} \setminus \{k\}$.*

Proof. (\Rightarrow) Let N be a classical prime submodule of M . We know $N = \times_{i=1}^n N_i$ where for every $1 \leq i \leq n$, N_i is a submodule of M_i , respectively. Assume that N_r is a proper submodule of M_r and N_s is a proper submodule of M_s for some $1 \leq r < s \leq n$. So, there are $m_r \in M_r \setminus N_r$ and $m_s \in M_s \setminus N_s$. Since

$$(0, \dots, 0, \overbrace{1_{R_r}}^{r\text{-th}}, 0, \dots, 0)(0, \dots, 0, \overbrace{1_{R_s}}^{s\text{-th}}, 0, \dots, 0)(0, \dots, 0, \overbrace{m_r}^{r\text{-th}}, 0, \dots, 0, \overbrace{m_s}^{s\text{-th}}, 0, \dots, 0)$$

$$= (0, \dots, 0) \in N,$$

then either

$$\begin{aligned} & (0, \dots, 0, \overbrace{1_{R_r}}^{r\text{-th}}, 0, \dots, 0)(0, \dots, 0, \overbrace{m_r}^{r\text{-th}}, 0, \dots, 0, \overbrace{m_s}^{s\text{-th}}, 0, \dots, 0) \\ &= (0, \dots, 0, \overbrace{m_r}^{r\text{-th}}, 0, \dots, 0) \in N \end{aligned}$$

or

$$\begin{aligned} & (0, \dots, 0, \overbrace{1_{R_s}}^{s\text{-th}}, 0, \dots, 0)(0, \dots, 0, \overbrace{m_r}^{r\text{-th}}, 0, \dots, 0, \overbrace{m_s}^{s\text{-th}}, 0, \dots, 0) \\ &= (0, \dots, 0, \overbrace{m_s}^{s\text{-th}}, 0, \dots, 0) \in N, \end{aligned}$$

which is a contradiction. Hence exactly one of the N_i 's is proper, say N_k . Now, we show that N_k is a classical prime submodule of M_k . Let $abm_k \in N_k$ for some $a, b \in R_k$ and $m_k \in M_k$. Therefore

$$\begin{aligned} & (0, \dots, 0, \overbrace{a}^{k\text{-th}}, 0, \dots, 0)(0, \dots, 0, \overbrace{b}^{k\text{-th}}, 0, \dots, 0)(0, \dots, 0, \overbrace{m_k}^{k\text{-th}}, 0, \dots, 0) \\ &= (0, \dots, 0, \overbrace{abm_k}^{k\text{-th}}, 0, \dots, 0) \in N, \end{aligned}$$

and so

$$(0, \dots, 0, \overbrace{a}^{k\text{-th}}, 0, \dots, 0)(0, \dots, 0, \overbrace{m_k}^{k\text{-th}}, 0, \dots, 0) = (0, \dots, 0, \overbrace{am_k}^{k\text{-th}}, 0, \dots, 0) \in N$$

or

$$(0, \dots, 0, \overbrace{b}^{k\text{-th}}, 0, \dots, 0)(0, \dots, 0, \overbrace{m_k}^{k\text{-th}}, 0, \dots, 0) = (0, \dots, 0, \overbrace{bm_k}^{k\text{-th}}, 0, \dots, 0) \in N.$$

Thus $am_k \in N_k$ or $bm_k \in N_k$ which implies that N_k is a classical prime submodule of M_k .

(\Leftarrow) Is easy. □

Theorem 2.27. *Let $R = R_1 \times R_2 \times \dots \times R_n$ ($2 \leq n < \infty$) be a decomposable ring and $M = M_1 \times M_2 \times \dots \times M_n$ be an R -module where for every $1 \leq i \leq n$, M_i is an R_i -module, respectively. For a proper submodule N of M the following conditions are equivalent:*

- (1) N is a classical 2-absorbing submodule of M ;
- (2) Either $N = \times_{t=1}^n N_t$ such that for some $k \in \{1, 2, \dots, n\}$, N_k is a classical 2-absorbing submodule of M_k , and $N_t = M_t$ for every $t \in \{1, 2, \dots, n\} \setminus \{k\}$ or $N = \times_{t=1}^n N_t$ such that for some $k, m \in \{1, 2, \dots, n\}$, N_k is a classical prime submodule of M_k , N_m is a classical prime submodule of M_m , and $N_t = M_t$ for every $t \in \{1, 2, \dots, n\} \setminus \{k, m\}$.

Proof. We argue induction on n . For $n = 2$ the result holds by Theorem 2.25. Then let $3 \leq n < \infty$ and suppose that the result is valid when $K = M_1 \times \dots \times M_{n-1}$. We show that the result holds when $M = K \times M_n$. By Theorem 2.25, N is a classical 2-absorbing submodule of M if and only if either $N = L \times M_n$ for some classical 2-absorbing submodule L of K or $N = K \times L_n$ for some classical 2-absorbing submodule L_n of M_n or $N = L \times L_n$ for some classical prime submodule

L of K and some classical prime submodule L_n of M_n . Notice that by Lemma 2.26, a proper submodule L of K is a classical prime submodule of K if and only if $L = \times_{t=1}^{n-1} N_t$ such that for some $k \in \{1, 2, \dots, n-1\}$, N_k is a classical prime submodule of M_k , and $N_t = M_t$ for every $t \in \{1, 2, \dots, n-1\} \setminus \{k\}$. Consequently we reach the claim. \square

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